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Waves and wave resistance of thin bodies moving at low speed: the free-surface nonlinear effect

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The linearized theory of free-surface gravity flow past submerged or floating bodies is based on a perturbation expansion of the velocity potential in the slenderness parameter ϵ with the Froude number F kept fixed. It is shown that, although the free-wave amplitude and the associated wave resistance tend to zero as $F \to 0$, the linearized solution is not uniform in this limit: the ratio between the second- and first-order terms becomes unbounded as $F \to 0$ with ϵ fixed. This non-uniformity (called 'the second Froude number paradox' in previous work) is related to the nonlinearity of the free-surface condition. Criteria for uniformity of the thin-body expansion, combining ϵ and F, are derived for two-dimensional flows. These criteria depend on the shape of the leading (and trailing) edge: as the shape becomes finer the linearized solution becomes valid for smaller F.

Uniform first-order approximations for two-dimensional flow past submerged bodies are derived with the aid of the method of co-ordinate straining. The straining leads to an apparent displacement of the most singular points of the body contour (the leading and trailing edges for a smooth shape) and, therefore, to an apparent change in the effective Froude number.

1. Introduction

The linearized theory of free-surface gravity flow past submerged or floating bodies is based on the assumption that the body causes a small disturbance to a uniform flow. Such an approximation is incorporated in a systematic asymptotic expansion of the velocity potential by assuming that ϵ (beam/length for a thin ship, draft/length for a flat ship, body length/submergence depth in the case of deep submergence) tends to zero while the Froude number F (based on body length or submergence depth, respectively) remains fixed.

In previous work (Salvesen 1969; Dagan 1972*a*) it has been shown that it is not legitimate to let $F \to 0$ for fixed ϵ in the linearized solution, or in other words that the usual approximation is not uniform in F. Two 'small Froude number paradoxes' have been formulated in this context (Dagan 1972*b*) and *ad hoc* uniformization procedures have been suggested (Ogilvie 1968; Dagan 1972*b*), leading to a quasi-linearization of the free-surface condition. It has been proved (Tuck 1965; Salvesen 1969; Dagan 1972*a*) that the small Froude number non-uniformity is associated with the nonlinearity of the free-surface condition. In all cases detailed computations have been carried out only for two-dimensional flows.

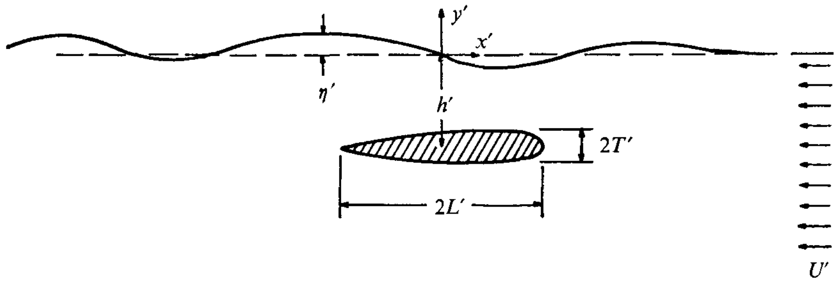

FIGURE 1

In the present study the problem of the small Froude number non-uniformity is attacked in a different way, and the influence of the bluntness of the bow on the small F solution is discussed in detail.

It is worthwhile to mention here that the problem is related mainly to three-dimensional applications, since a large class of ships operate at relatively low Froude numbers and in most such cases the usual theory of wave resistance has been found to be unsatisfactory. We carry out, nevertheless, the study of the two-dimensional flow because the use of the powerful tool of analytical functions in this case permits us to clarify some matters of principle much more easily than we could in three dimensions.

Obviously, there are various factors which may be related to the discrepancy between the wave resistance measured in experiments and that predicted by the linearized theory, like viscous effects or the bow breaking wave. This should not deter us, however, from seeking a consistent solution for the wave resistance within the framework of potential flow theory.

2. Two-dimensional flow past submerged bodies

2.1. *The thin-body expansion*

We consider a steady uniform flow from infinity past a submerged body (figure 1). Let $z' = x' + iy'$ be a complex variable, $w' = u' - iv'$ the complex velocity, $f' = \phi' + i\psi'$ the complex potential, η' the free-surface elevation, $2L'$ the body length, h' the submergence depth, $2T'$ the body thickness and U' the velocity of the uniform flow. First, variables are made dimensionless by referring them to L' and U', i.e. $z = z'/L'$, $f = f'/L'U'$, $w = w'/U'$, $h = h'/L'$, $\epsilon = T'/L'$ and $F = U'/(gL')^{\frac{1}{2}}$.

If the analytical function $f(z; \epsilon, h, F)$ is expanded in an asymptotic series

$$f = -z + \epsilon f_1(z; h, F) + \epsilon^2 f_2(z; h, F) + \dots \tag{1}$$

for small ϵ, the following sets of equations are obtained for f_1 and f_2 from the expansion of the exact equations (Wehausen & Laitone 1960):

$$\left.\begin{aligned} \operatorname{Im}(iF^2\, df_1/dz - f_1) &= 0 \\ \eta_1 &= \psi_1 \end{aligned}\right\} \quad (y = 0), \tag{2), (3}$$

$$f_1 \to 0 \quad (x \to \infty,\ y \to -\infty), \tag{4}$$

$$\psi_1(x, -h+0) - \psi_1(x, -h-0) = -2t(x) \quad (|x| < 1), \tag{5}$$

where $\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots$ and $y = -h \pm \epsilon t(x)$ is the equation of the body profile, assumed to be symmetrical for the sake of simplicity;

$$\left.\begin{array}{r}\operatorname{Im}(iF^2\,df_2/dz - f_2) + P_2(x) = F^2(\tfrac{3}{2}u_1^2 + \tfrac{1}{2}v_1^2) - F^4u_1\,\partial v_1/\partial x \\ \eta_2 = \psi_2 + \psi_1 u_1\end{array}\right\} \quad (y = 0), \qquad (6), (7)$$

$$f_2 \to 0 \quad (x \to \infty,\ y \to -\infty), \qquad (8)$$

$$\psi_2(x, -h+0) - \psi_2(x, -h-0) = -2u_1 t \quad (|x| < 1,\ y = -h). \qquad (9)$$

In addition, a Kutta–Joukowsky condition must be imposed in the case of a sharp trailing edge in order to make the circulation unique.

It can be shown (Salvesen 1969) that far behind the body the stream function contains the expressions

$$\psi_1 = \operatorname{Im}(a_1 e^{-ix}) \qquad (x \to -\infty), \qquad (10)$$

$$\psi_2 = \operatorname{Im}(a_2 e^{-ix}) + \text{constant} \quad (x \to -\infty). \qquad (11)$$

The wave resistance is given by

$$D = (2F^2)^{-1}|\epsilon a_1 + \epsilon^2 a_2|^2, \qquad (12)$$

where $D = D'/\tfrac{1}{2}\rho' U'^2 L'$. Hence, by expanding D as

$$D = \epsilon^2 D_1 + \epsilon^3 D_2 + O(\epsilon^4) \qquad (13)$$

we have

$$D_1 = (2F^2)^{-1}|a_1|^2, \quad D_2 = F^{-2}\operatorname{Re}(a_1\bar{a}_2). \qquad (14)$$

The method of determining f_1 and f_2, the solutions of (2)–(8), is well known. Let w_1^l and w_1^u be the first-order linearized solutions for the velocity of flow past the body and its image, respectively, in an infinite domain, i.e.

$$w_1^l(z) = -\frac{1}{\pi}\int_{-1-ih}^{1-ih} \tau(x_s)\frac{dz_s}{z - z_s}, \qquad (15)$$

where $z_s = x_s + iy_s$ ($|x_s| < 1$, $y_s = -h$) is the co-ordinate of a point along the body contour,

$$\operatorname{Re}\tau = dt/dx$$

and

$$w_1^u(z) = -\frac{1}{\pi}\int_{-1+ih}^{1+ih} \bar{\tau}(x_s)\frac{d\bar{z}_s}{z - \bar{z}_s} = \overline{w}_1^l(\bar{z}). \qquad (16)$$

Then the solution for f_1 may be written as

$$f_1 = f_1^l + f_1^u - \frac{i}{\pi}\int_{-\infty}^{\infty} w_1^u(\sigma)\,\omega\!\left(\frac{z-\sigma}{F^2}\right)d\sigma \quad (\operatorname{Im} z < \operatorname{Im}\sigma), \qquad (17)$$

where

$$\omega(\zeta) = e^{-i\zeta}\int_{\zeta}^{\infty - i0}\frac{e^{i\lambda}}{\lambda}\,d\lambda = \int_0^{\infty}\frac{e^{i\rho}}{\rho + \zeta}\,d\rho, \qquad (18)$$

the λ plane being cut along $\operatorname{Im}\lambda = 0$, $\operatorname{Re}\lambda > 0$, while the ρ plane is cut along $\operatorname{Im}(\rho + \zeta) = 0$, $\operatorname{Re}(\rho + \zeta) > 0$. The function ω in (17) is regular in the upper half σ plane while w_1^u is singular along the slit $|x| < 1$, $y = h$. Hence the integration in (17) may be replaced by an integral around the slit which leaves in (17) only the jump $2\tau(x_s)$ in the imaginary part of w_1^u.

The second-order solution satisfying (6) and regular in the lower half-plane may be written as

$$f_2(z) = \frac{i}{\pi F^2}\int_{-\infty}^{\infty} P_2(\sigma)\,\omega\!\left(\frac{z-\sigma}{F}\right)d\sigma \quad (\operatorname{Im} z < \operatorname{Im}\sigma). \qquad (19)$$

We disregard here the contribution of the body correction term, related to (9), as well as the contribution of the vorticity, related to the cross-flow induced by the image singularities of f_1 upon the body contour, because these terms lead to less singular contributions to f_2, as $F \to 0$, than the free-surface condition.

Circulation associated with the Kutta–Joukowsky condition at a sharp trailing edge of a body of finite thickness may play an important role (Salvesen 1969). However, as the present two-dimensional solution serves only as a tool for a better understanding of the thin-ship problem, we consider here solely the thickness effect.

2.2. *The second-order solution (free-surface effect)*

We are going now to transform (19) such that f_2 may be expressed as an integral over analytical functions of σ. First, we have, by integration by parts,

$$\int_{-\infty}^{\infty} u_1 \frac{\partial v_1}{\partial \sigma} \omega\left(\frac{z-\sigma}{F^2}\right) d\sigma = -\frac{i}{F^2}\int_{-\infty}^{\infty} (u_1 v_1 + i v_1)\, \omega\left(\frac{z-\sigma}{F^2}\right) d\sigma - \int_{-\infty}^{\infty} \frac{u_1 v_1}{z-\sigma}\, d\sigma. \tag{20}$$

u_1 and v_1, which are obtained from (17), may be written along the real axis as

$$u_1(x) = -\frac{1}{2\pi F^2}\int_{-\infty}^{\infty} \left[w_1^u(\tau)\, \omega\left(\frac{x-i0-\tau}{F^2}\right) + w_1^l(\tau)\, \overline{\omega}\left(\frac{x+i0-\tau}{F^2}\right)\right] d\tau, \tag{21}$$

$$v_1(x) = i[w_1^l(x) - w_1^u(x)] - \frac{i}{2\pi F^2}\int_{-\infty}^{\infty} \left[w_1^u(\tau)\, \omega\left(\frac{x-i0-\tau}{F^2}\right) - w_1^l(\tau)\, \overline{\omega}\left(\frac{x+i0-\tau}{F^2}\right)\right] d\tau, \tag{22}$$

where $\overline{\omega}$ is defined [cf. (18)] as

$$\overline{\omega}(\zeta) = \int_0^{\infty} \frac{e^{-i\lambda}}{\lambda+\zeta}\, d\lambda. \tag{23}$$

Substituting (21) and (22) in (20) and integrating by parts we obtain f_2 (for details see appendix) in its final form as

$$f_2(z) \cong -\frac{i}{\pi}\int_{-\infty}^{\infty} \left\{ -\tfrac{1}{2}[w_1^l(\sigma) - w_1^u(\sigma)]^2 + \frac{1}{\pi F^2}[w_1^l(\sigma) + w_1^u(\sigma)] \int_{-\infty}^{\infty} w_1^u(\tau)\, \omega\left(\frac{\sigma-\tau}{F^2}\right) d\tau \right\} \omega\left(\frac{z-\sigma}{F^2}\right) d\sigma + \frac{F^2}{2\pi}\int \frac{(w_1 - w_1^l - w_1^u)^2}{z-\sigma}\, d\sigma, \tag{24}$$

where terms which tend to zero for $x \to -\infty$ have been neglected.

2.3. *Illustration of results: the source–sink body*

The complete solution. Rather than pursuing a general discussion of the second-order solution in the limit $F \to 0$, we begin with a simple case which can be solved in a closed form.

We consider the closed body generated by a source at $z_l = 1 - ih$ and a sink at $z_t = -1 - ih$. In view of our interest in three-dimensional applications we consider only the thickness effect. This is the first-order representation of a straight thin body with blunt leading and trailing edges.

Following now (15)–(17) and (24) we have

$$f_1 = \frac{1}{2\pi}\ln\frac{z-z_l}{z-z_t} + \frac{1}{\pi}\,\omega\left(\frac{z-\bar{z}_l}{F^2}\right) - \frac{1}{\pi}\,\omega\left(\frac{z-\bar{z}_t}{F^2}\right), \tag{25}$$

$$w_1^l = \frac{1}{2\pi}\frac{1}{z-z_l} - \frac{1}{2\pi}\frac{1}{z-z_t}, \quad w_1^u = \frac{1}{2\pi}\frac{1}{z-\bar{z}_l} - \frac{1}{2\pi}\frac{1}{z-\bar{z}_t}, \tag{26), (27}$$

$$f_2 \cong -\frac{i}{4\pi^3}\int_{-\infty}^{\infty}\left\{-\frac{1}{2}\left[\frac{1}{\sigma-z_l} - \frac{1}{\sigma-z_t} - \frac{1}{\sigma-\bar{z}_l} + \frac{1}{\sigma-\bar{z}_t}\right]^2 + \frac{1}{\pi F^2}\left(\frac{1}{\sigma-z_l} - \frac{1}{\sigma-z_t} + \frac{1}{\sigma-\bar{z}_l} - \frac{1}{\sigma-\bar{z}_t}\right)\int_{-\infty}^{\infty}\left(\frac{1}{\rho-\bar{z}_l} - \frac{1}{\rho-\bar{z}_t}\right) \times \omega\left(\frac{\sigma-\rho}{F^2}\right)d\rho\right\}\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma - \frac{1}{2\pi^3F^2}\int_{-\infty}^{\infty}\frac{\{\omega[(\sigma-\bar{z}_l)/F^2] - \omega[(\sigma-\bar{z}_t)/F^2]\}^2}{z-\sigma}\,d\sigma. \tag{28}$$

The integration in (28) can be carried out exactly. The first term, resulting from $w_1^l - w_1^u$, contributes the residues at $\sigma = \bar{z}_l$ and $\sigma = \bar{z}_t$. The last terms are more intricate, but still tractable, at least for $x \to -\infty$.

We consider now the expansions (25) and (28) of f_1 and f_2 for small F.

The near-field solution. For $F^2 \to 0$, $\omega[(z-\bar{z}_l)/F^2]$ can be expanded in an asymptotic series for fixed z as follows:

$$\omega\left(\frac{z-\bar{z}_l}{F^2}\right) \sim -\sum_{n=1}^{\infty}\frac{(n-1)!\,F^{2n}}{i^n(z-\bar{z}_l)^n}. \tag{29}$$

This expansion is valid, however, only for $|z-\bar{z}_l| > \delta$ and $-\pi+\bar{\delta} < \arg(z-\bar{z}_l) < 0$, where δ and $\bar{\delta}$ are arbitrarily small fixed quantities (for details see the discussion of the related exponential integral function in Copson 1965, p. 25).

Substitution of (29) and the similar expansion of $\omega[(z-\bar{z}_t)/F^2]$ into w_1, obtained by differentiation of f_1, as given by (25), yields

$$w_1 = w_1^l + w_1^u + O(F^2) \quad (-\pi+\bar{\delta} < \arg(z-\bar{z}_1) < 0). \tag{30}$$

Hence, w_1 degenerates at zero order into the rigid-wall solution, i.e. the solution for flow past the body in the presence of a rigid wall at $y = 0$. However, this limit is not uniform and in particular is not valid far behind the body, i.e. for $x \to -\infty$ with y kept fixed. For $\arg(z-\bar{z}_1) > -\pi+\bar{\delta}$ expansion (29) has to be supplemented by the term $2\pi i \exp(\pm\bar{z}_l/F^2)\exp(-iz/F^2)$, which represents precisely the trailing waves. For this reason (30) may be called the near-field expansion.

The rigid-wall solution, and the subsequent terms of (30), may be obtained also by first expanding the linearized free-surface condition (2) for $F^2 \to 0$ and then solving term by term. In contrast with the previous procedure, however, (30) is thus obtained as a solution uniformly valid in the *entire z plane*. Although the wave term is exponentially small for $y < h$, compared with the powers of F^2 in (30), it is the only one which does not tend to zero for $x \to -\infty$, y fixed and which is associated with wave resistance.

Similarly, the near-field expansion of $f_2(z)$ for $F^2 \to 0$ may be obtained from (28) by expanding $\omega[(z-\sigma)/F^2]$ and computing the residues at $\bar{z}_l$ and $\bar{z}_t$. The result is $O(F^2)$ and is a rational function of z with poles of different orders at $z = \bar{z}_l$ and $z = \bar{z}_t$. Hence, ϵf_1 is $O(\epsilon)$ while $\epsilon^2 f_2$ is $O(\epsilon^2 F^2)$ and the near-field expansion of f is *uniform* as $F^2 \to 0$.

The free-wave potential. The free-wave potential is obtained from (25) and (28) by letting $x \to -\infty$. The first-order solution (25) yields

$$f_1^w = 2i[\exp(i\bar{z}_l/F^2) - \exp(i\bar{z}_t/F^2)]\exp(-iz/F^2)$$
$$= -4(\sin F^{-2})\exp(-h/F^2)\exp(-iz/F^2). \quad (31)$$

In the second-order solution (28) $\omega[(z-\sigma)/F^2]$ is first replaced by

$$2\pi i \exp(-iz/F^2)\exp(i\sigma/F^2).$$

Integration then yields (for details see Dagan 1973)

$$f_2^w \cong +F^{-2}[(b_2+ic_2)\exp(i\bar{z}_l/F^2)$$
$$+(b_2-ic_2)\exp(i\bar{z}_t/F^2)]\exp(-iz/F^2)+O(\exp(-h/F^2),\exp(-2h/F^2)), \quad (32)$$

where $$b_2 = \frac{1}{\pi}\left(\tfrac{1}{2}+2C+\ln 4+\ln\frac{1+h^2}{h^2}\right), \quad c_2 = \frac{1}{\pi}\left(\frac{\pi}{2}-\arctan\frac{1}{h}\right)$$

and C is Euler's constant. If $h \ll 1$, $c_2 \simeq 0$ and (32) becomes

$$f_2^w \simeq +\frac{2b_2}{F^2}\left(\cos\frac{1}{F^2}\right)\exp\left(\frac{-h}{F^2}\right)\exp\left(\frac{-iz}{F^2}\right)+O\left(\exp\left(\frac{-h}{F^2}\right)\exp\left(\frac{-2h}{F^2}\right)\right). \quad (33)$$

Hence, the amplitude of the free waves, by (10), (11), (31) and (32), is given by

$$\epsilon a_1 = O(\epsilon\exp(-h/F^2)), \quad (34)$$
$$\epsilon^2 a_2 = O(\epsilon^2\exp(-h/F^2)/F^2), \quad (35)$$

and although for $F^2 \to 0$ with h and ϵ fixed both ϵa_1 and $\epsilon^2 a_2$ tend to zero, their ratio becomes unbounded like ϵ/F^2.

This non-uniformity of the thin-body expansion has been described previously by Salvesen (1969). Equation (35) shows that the usual linearized theory is valid, for the source–sink body, only if $\epsilon/F^2 = O(1)$, i.e. for large Froude numbers based on thickness.

2.4. *Generalization for bodies of different shapes*

Since for an arbitrary thickness distribution w_1^l and w_1^u are represented by the source distributions (15) and (16) the results of the previous section may be extended to thin bodies of any shape. It is easy to ascertain that the near-field solution, based on (17) and (29), has the rigid-wall approximation as a leading term and is uniform in the sector $\pi-\delta < \arg(z-1-ih) < 0$ as $F^2 \to 0$.

The non-uniformity of the expansion for the free waves depends essentially on the bluntness of the leading edge (for the sake of simplicity we consider bodies of smooth shape and assume that viscous effects ensure anyway that the trailing edge has a fine shape). The free waves are represented at first order by

$$f_1^w = 2\exp(-iz/F^2)\oint w_1^u(\sigma)\exp(i\sigma/F^2)\,d\sigma, \quad (36)$$

which has been obtained from (17), the integration path circumventing the contour of the image of the body $-1 < \sigma - ih < 1$ in the upper half-plane. For

Shape of the leading edge	The singularity of w_1^u	Order of a_1 for $F \to 0$	Order of a_2 for $F \to 0$	Order of $\epsilon^2 a_2/\epsilon a_1$	Order of the straining $\delta \bar{z}_l$
$\supset$	$(z-1-ih)^{-1}$	$\exp(-h/F^2)$	$\exp(-h/F^2)/F^2$	ϵ/F^2	ϵ
)	$(z-1-ih)^{\frac{1}{2}}$	$F\exp(-h/F^2)$	$\exp(-h/F^2)$	ϵ/F	ϵF
>	$\ln(z-1-ih)$	$F^2(\ln F)\exp(-h/F^2)$	$\exp(-h/F^2)F^2\ln^2 F$	$\epsilon \ln F$	$\epsilon F^2 \ln F$

TABLE 1

$F^2 \to 0$ the integral in (36) may be expanded in the usual manner, the lowest-order term being provided by the highest singularities of $w_1^u(\sigma)$, those at $\sigma = \pm 1 + ih$.

We have seen that for a source-like blunt shape $a_1 = O(\exp(-h/F^2))$. For a leading edge of elliptical shape (i.e. $w_1^u \sim 1/(\sigma-1-ih)^{\frac{1}{2}}$), (36) shows that $a_1 = O(F\exp(-h/F^2))$. Similarly, for a wedge-like shape ($w_1^u \sim \ln(\sigma-1-ih)$) we obtain $a_1 = O(F^2 \ln F \exp(-h/F^2))$ (see Lighthill 1964, p. 43).

To estimate the order of the amplitude of the free waves at second order in ϵ we must use the expression (24) for f_2 with $\omega[(z-\sigma)/F^2]$ replaced by

$$2\pi i \exp(-iz/F^2)\exp(i\sigma/F^2).$$

The computation is facilitated by the observation, supported by the detailed solution of the previous section, that the order of the lowest-order term in F is determined by the term $[w_1^u(\sigma)]^2$ in the integral for $f_2(z)$ in (24), the other terms contributing at an equal or higher order. Hence, the order of f_1^w is determined by integrals of the type

$$\exp(-iz/F^2)\oint [w_1^u(\sigma)]^2 \exp(i\sigma/F^2)\, d\sigma. \qquad (37)$$

We have, therefore, for an elliptical leading edge $a_2 = O(\exp(-h/F^2))$ and for a wedge-like shape $a_2 = O(F^2 \ln^2 F \exp(-h/F^2))$. In each case the far-wave amplitude, and consequently the wave resistance, is not uniform for $F^2 \to 0$, the non-uniformity becoming, however, weaker as the shape of the edge becomes finer. The results are collected in table 1. The last column but one summarizes the main findings: the quantity appearing there has to be small in order to ensure that the usual linearized thin-body approximation is uniform. It is worthwhile mentioning that in all the examples for which detailed computations have been carried out so far (Tuck 1965, for a circular cylinder; Salvesen 1969, for a hydrofoil; Dagan 1972*a*, for a source), the shapes were blunt.

2.5. *Derivation of uniform small Froude number solutions*

Procedures for rendering the first-order solution uniform have been suggested previously by Ogilvie (1968) and Dagan (1972*b*) (in both cases the more ambitious task of solving the problem of small Froude number flow past a body of finite thickness was undertaken). The constant coefficient of df_1/dz in (2) was replaced by a variable one, equal to the velocity of the first-order rigid-wall solution at $y = 0$. Although the solution obtained this way is in principle uniform, it can be shown (Dagan 1973) that additional terms, besides the rigid-wall solution, have to be incorporated in the coefficient of df_1/dz in (2). Moreover, the procedure becomes extremely tedious in the case of three-dimensional flows (Dagan 1973). An alternative method, co-ordinate straining, is presented here.

We assume that the small Froude number non-uniformity is a result of co-ordinate straining. Lighthill's method (see Van Dyke 1964, p. 99) implies an infinitesimal straining of the physical plane and derives the straining function from the equations of flow. We adopt here a modified technique applied by Van Dyke (1964, p. 72) to the case of inviscid flow past airfoils: we carry out the straining in the solution, rather than in the equations, and determine the straining function from the requirement that the second-order term should not be more singular than the first-order term. In the case of airfoils the solution becomes singular as the leading edge is approached. In our case the free-wave potential is not uniform as $F^2 \to 0$; the two problems are therefore quite different.

We consider the change of variables

$$z = \zeta + \delta z(\zeta). \tag{38}$$

The first-order potential (36) of the free waves becomes, in terms of the strained co-ordinate ζ,

$$f_1^w(\zeta) = -\frac{2}{\pi}\exp(-i\zeta/F^2)\int_{-\infty}^{\infty} d\sigma \exp(i\sigma/F^2)\int_{\bar{\zeta}_t}^{\bar{\zeta}_l}\frac{\tau(\bar{\zeta}_s)}{\sigma-\bar{\zeta}_s}\,d\bar{\zeta}_s \quad (\zeta\to-\infty), \tag{39}$$

where $\zeta = \bar{\zeta}_s = \bar{z}_s - \delta\bar{z}_s$ is the mapping of the image of the body axis $z = \bar{z}_s = x_s + ih$ onto the ζ plane and $\bar{\zeta}_t$ and $\bar{\zeta}_l$ map the co-ordinates of the trailing and leading edges, $\bar{z}_t = -1+ih$ and $\bar{z}_l = 1+ih$, respectively ($\bar{\zeta}_l = \bar{z}_l - \delta\bar{z}_l$, $\bar{\zeta}_t = \bar{z}_t - \delta\bar{z}_t$).

To determine the straining $\delta z(\zeta) = O(\epsilon)$ we first expand the expression (39) for f_1^w for F fixed and $\epsilon = o(1)$. In such an expansion terms of order ϵ in (39) will result from the limits of the second integral, from $\tau(\bar{\zeta}_s)$ and from the expansion of $\bar{\zeta}_s$ in the denominator of the integrand. But only the latter expansion provides a term $O(\epsilon/F^2)$, regardless of whether the first two lead to terms $O(\epsilon)$. Hence, the lowest-order terms for $F\to 0$ are provided by

$$f_1^w(z) \simeq -\frac{2}{\pi}\exp(-iz/F^2)\int_{-\infty}^{\infty}\exp(i\sigma/F)\,d\sigma\int_{-1}^{1}\frac{\tau(x_s)}{\sigma-\bar{z}_s+\delta\bar{z}_s}\,dx_s. \tag{40}$$

In other words the straining is manifest solely in a virtual displacement of the body singularities, with no change in their strength, as far as most singular terms (for $F\to 0$) are concerned.

For $\delta\bar{z}_s = O(\epsilon)$ and F fixed the integrand in (40) may be expanded as follows:

$$\begin{aligned} f^w &= -\frac{2\epsilon}{\pi}\exp\left(-\frac{iz}{F^2}\right)\int_{-\infty}^{\infty} d\sigma\int_{-1}^{1} dx_s\frac{\tau}{\sigma-\bar{z}_s}\exp\left(\frac{i\sigma}{F^2}\right) \\ &\qquad + \frac{2\epsilon}{\pi}\exp\left(-\frac{iz}{F^2}\right)\int_{-\infty}^{\infty} d\sigma\exp\left(\frac{i\sigma}{F^2}\right)\int_{-1}^{1} dx_s\frac{\tau\delta\bar{z}_s}{(\sigma-\bar{z}_s)^2} \\ &= -4i\epsilon\exp\left(-\frac{iz}{F^2}\right)\int_{-1}^{1}\tau(x_s)\exp\left(\frac{i\bar{z}}{F^2}\right)dx_s \\ &\qquad -\frac{4\epsilon}{F^2}\exp\left(-\frac{iz}{F^2}\right)\int_{-1}^{1}\tau(x_s)\,\delta\bar{z}_s\exp\left(\frac{i\bar{z}_s}{F^2}\right)dx_s. \end{aligned} \tag{41}$$

The first term in (41) is precisely f_1^w [see (36)]; the unknown straining function $\delta\bar{z}_s$ is now determined from the requirement that the last term of (41) should cancel the term of lowest order in F of f_2^w [see (24)].

To determine $\delta\bar{z}_s$ in a simple way, advantage is taken of the fact that the terms of lowest order in F in (41) are associated with the singularities at the edges (an intermediate point of discontinuity can be easily accounted for). What matters, therefore, is $\delta\bar{z}_l$ and $\delta\bar{z}_t$. Any continuous straining between the edges is acceptable as far as the most singular terms are concerned. Assuming, for the sake of simplicity, a linear straining we have

$$\delta\bar{z}_s = \tfrac{1}{2}(\delta\bar{z}_l - \delta\bar{z}_t)x_s + \tfrac{1}{2}(\delta\bar{z}_l + \delta\bar{z}_t). \tag{42}$$

Substitution of (42) into the last term of (41) yields for the terms of lowest order in F

$$-\frac{2\epsilon}{F^2}\exp(-iz/F^2)(\delta\bar{z}_l + \delta\bar{z}_t)\int_{-1}^{1}\tau(x_s)\exp(i\bar{z}_s/F^2)\,dx_s. \tag{43}$$

Equating to zero the sum of the terms of lowest order in F in f_2^w [see (24) using (43)] gives a unique expression for $\delta\bar{z}_l + \delta\bar{z}_t$. An additional relationship is obtained from the requirement of separate cancellation of the leading- and trailing-edge waves (obviously, for a fine trailing edge, $\delta\bar{z}_t = 0$).

To illustrate the method we consider the example of a source–sink body [see §2.3). The uniform first-order term of the free-wave expansion becomes for $x \to -\infty$, by using (17), (26), (27) and (38),

$$f^w = \frac{\epsilon}{\pi}\exp\left(-\frac{iz}{F^2}\right)\int_{-\infty}^{\infty}\left(\frac{1}{\sigma - \bar{z}_l + \delta\bar{z}_l} - \frac{1}{\sigma - \bar{z}_t + \delta\bar{z}_t}\right)\exp\left(\frac{i\sigma}{F^2}\right)d\sigma. \tag{44}$$

Expanding, as in (41), for F fixed and $\epsilon \to 0$, we obtain

$$\begin{aligned} f^w &= \frac{\epsilon}{\pi}\exp\left(-\frac{iz}{F^2}\right)\int_{-\infty}^{\infty}\left(\frac{1}{\sigma - \bar{z}_l} - \frac{1}{\sigma - \bar{z}_t}\right)\exp\left(\frac{i\sigma}{F^2}\right)d\sigma \\ &\quad - \frac{\epsilon}{\pi}\exp\left(-\frac{iz}{F^2}\right)\int_{-\infty}^{\infty}\left[\frac{\delta\bar{z}_l}{(\sigma - \bar{z}_l)^2} - \frac{\delta\bar{z}_t}{(\sigma - \bar{z}_t)^2}\right]\exp\left(\frac{i\sigma}{F^2}\right)d\sigma \\ &= f_1^w - \frac{2\epsilon}{F^2}\exp\left(-\frac{iz}{F^2}\right)\left(\delta\bar{z}_l\exp\left(\frac{i\bar{z}_l}{F^2}\right) - \delta\bar{z}_t\exp\left(\frac{i\bar{z}_t}{F^2}\right)\right). \end{aligned} \tag{45}$$

Hence, the first term of (45) is f_1^w [see (31)]. Consequently, the second-order term of the free-wave potential will be made up this time of f_2^w [see (32)] plus the last term of (45), provided that the straining is of order ϵ.

We now determine $\delta\bar{z}_l$ and $\delta\bar{z}_t$ from the requirement that the term of order ϵ/F^2 in the amplitude of the free waves, which is the origin of the small F non-uniformity, should vanish in the solution, separately for the source and the sink. We thus obtain

$$\delta\bar{z}_l = (-\epsilon/2\pi)(b_2 + ic_2), \tag{46}$$

$$\delta\bar{z}_t = (+\epsilon/2\pi)(b_2 - ic_2), \tag{47}$$

where b_2 and c_2 are given in (32). The uniform first-order solution, valid for $\epsilon/F^2 = O(1)$, is easily derived from (44):

$$f_1^w = 2i\exp(-iz/F^2)\{\exp[i(\bar{z}_l - \delta\bar{z}_l/F^2] - \exp[i(\bar{z}_t - \delta\bar{z}_t)/F^2]\}. \tag{48}$$

By using (46) and (47) we finally obtain from (48)

$$f_1^w = -4\exp\left(\frac{-[h+c_2\epsilon/2\pi]}{F^2}\right)\sin\frac{1+\epsilon b_2/2\pi}{F^2}\exp\left(-\frac{iz}{F^2}\right). \tag{49}$$

Hence the straining has the effect of an apparent *increase* of the length of the body by a factor of $1+\epsilon b_2/2\pi$. The virtual increase in the submersion depth is negligible for $h \ll 1$ [see (32)].

In the general case (40) the estimates of §2.4 permit evaluation of the order of the straining $\delta \bar{z}_l$ for different types of leading-edge singularities. The results are given in the last column of table 1. The straining becomes weak as the shape of the leading edge becomes fine, and it is F dependent except in the source-like case.

2.6. *Conclusions*

It has been shown that the slenderness parameter ϵ appears in the expression for the potential of the free waves generated by a submerged body not only in the amplitude, but also as the ratio ϵ/F^2 in the wavenumber. As in other problems characterized by two scales (Cole 1968), a power-series expansion in ϵ does not yield a uniform solution unless $\epsilon/F^2 = o(1)$; this last estimate has been sharpened and shown to depend on the nature of the singularity at the leading (and trailing) edge.

The non-uniformity of the thin-body expansion may be interpreted in relation to the linearized pressures $\epsilon P_1(x)$ [see (17)] and $\epsilon^2 P_2(x)$ [see (6)]: although the ratio of the amplitudes tends to zero as $\epsilon \to 0$, $P_2(x)$ is more oscillatory than $P_1(x)$ and the amplitude of the system of free waves associated with it is amplified as $F^2 \to 0$. A more convenient interpretation is reached from inspection of the free-wave potentials as expressed by integrals along the image of the body contour, (36) and (37). The first-order free waves f_1^w are generated by the singularities of the velocity w_1^u of flow in an infinite domain [see (36)]; as the wavelength tends to zero the largest contributions originate from the points of largest slope of the profile, i.e. from the leading and trailing edges. At second order the free waves are associated with $(w_1^u)^2$ [see (37)], i.e. with a distribution which is more singular at the edges and which generates waves of higher amplitude depending on the singularity and on the wavelength.

It is worthwhile to emphasize that the nonlinear effects considered here are essentially associated with the local disturbance pressure $P_2(x)$ and not with nonlinear interaction between the far free waves.

As in the case of the airfoil, the method of co-ordinate straining suggests that the worsening of the singular behaviour at the edges may be removed by an infinitesimal displacement of the most singular points of the profile. When the wavelength of the free waves becomes of the same order as the straining, the amplitude is profoundly influenced by the straining. The straining leads to a virtual forward displacement (49) of the leading edge in the expression for the free waves and, correspondingly, to a shift of the resistance curve. Obviously, a power-series expansion in ϵ of f^w [see (49)], as implied by the usual thin-body theory, is legitimate only if $\epsilon/F^2 = o(1)$, or under other criteria for finer shapes (table 1).

The analysis has been extended to three-dimensional flow past thin ships (Dagan 1973). Although the computations become much more complex, the results are similar to those presented here. For instance, in the case of a wedge-like bow the first- and second-order amplitudes of the free waves are $O(\epsilon F^2)$ and $O(\epsilon^2 F^2 \ln F)$, respectively.

A more general approach to co-ordinate straining, which leads to solutions satisfying the free-surface and body conditions to second order at any Froude number, has been presented in two recent studies (Noblesse 1975; Dagan 1975).

Appendix. Derivation of $f_2(z)$ [see (24)]

The expression (20) for f_2 becomes, by substitution of u_1 and v_1 from (21) and (22),

$$f_2(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\sigma\, \omega\left(\frac{z-\sigma}{F^2}\right)\left\{-\tfrac{1}{2}[w_1^l(\sigma)-w_1^u(\sigma)]^2 - \frac{1}{\pi F^2}[w_1^l(\sigma)-w_1(\sigma)]\right.$$
$$\times\left[\int_{-\infty}^{\infty} d\tau\, w_1^l(\tau)\,\overline{\omega}\left(\frac{\sigma-\tau}{F^2}\right)\right] - \frac{1}{4\pi^2F^4}\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2\left[w_1^u(\tau_1)\,w_1^u(\tau_2)\right.$$
$$\times\,\omega\left(\frac{\sigma-\tau_1}{F^2}\right)\omega\left(\frac{\sigma-\tau_2}{F^2}\right) + 2w_1^l(\tau_1)\,w_1^u(\tau_2)\,\overline{\omega}\left(\frac{\sigma-\tau_1}{F^2}\right)\omega\left(\frac{\sigma-\tau_2}{F^2}\right)$$
$$\left.\left. + 3w_1^l(\tau_1)\,w_1^l(\tau_2)\,\overline{\omega}\left(\frac{\sigma-\tau_1}{F^2}\right)\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\right]\right\}. \quad \text{(A 1)}$$

By integration by parts it can be shown that

$$\int_{-\infty}^{\infty}\omega\left(\frac{\sigma-\tau_1}{F^2}\right)\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma = -iF^2\int_{-\infty}^{\infty}\left[\frac{1}{\sigma-\tau_2}\,\omega\left(\frac{\sigma-\tau_1}{F^2}\right)\right.$$
$$\left.+\frac{1}{\sigma-\tau_1}\,\omega\left(\frac{\sigma-\tau_2}{F^2}\right)\right]\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma - iF^2\int_{-\infty}^{\infty}\omega\left(\frac{\sigma-\tau_1}{F^2}\right)\omega\left(\frac{\sigma-\tau_2}{F^2}\right)\frac{1}{z-\sigma}\,d\sigma, \quad \text{(A 2)}$$

$$\int_{-\infty}^{\infty}\omega\left(\frac{\sigma-\tau_1}{F^2}\right)\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma = -iF\int_{-\infty}^{\infty}\left[\frac{1}{\sigma-\tau_2}\,\omega\left(\frac{\sigma-\tau_1}{F^2}\right)\right.$$
$$\left.+\frac{1}{\sigma-\tau_1}\,\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\right]\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma + iF^2\int_{-\infty}^{\infty}\omega\left(\frac{\sigma-\tau_1}{F^2}\right)\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\frac{1}{z-\sigma}\,d\sigma, \quad \text{(A 3)}$$

$$3\int_{-\infty}^{\infty}\overline{\omega}\left(\frac{\sigma-\tau_1}{F^2}\right)\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma = iF^2\int_{-\infty}^{\infty}\overline{\omega}\left(\frac{\sigma-\tau_1}{F^2}\right)\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\frac{1}{z-\sigma}\,d\sigma$$
$$-iF^2\int_{-\infty}^{\infty}\left[\overline{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)\frac{1}{\sigma-\tau_1}+\overline{\omega}\left(\frac{\sigma-\tau}{F^2}\right)\frac{1}{\sigma-\tau_2}\right]\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma. \quad \text{(A 4)}$$

By residues we also have

$$\int_{-\infty}^{\infty}\frac{w_1^u(\tau_2)}{\sigma-i0-\tau_2}\,d\tau_2 = 2\pi i w_1^u(\sigma), \quad \int_{-\infty}^{\infty}\frac{w_1^l(\tau_1)}{\sigma+i0-\tau_1}\,d\tau_1 = -2\pi i w_1^l(\sigma). \quad \text{(A 5)}$$

Substituting (A 2), (A 3) and (A 4) in (A 1) results in the final expression (24) for $f_2(z)$.

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